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Reduction of the equations of dynamics of systems with constraints to a given structure☆

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Abstract

The problem of constructing systems of second-order ordinary differential equations, the solutions of which, with the appropriate initial conditions, satisfy given equations of the constraints, is considered. The conditions for representing the differential equations in the form of Lagrange equations of the second kind are determined. It is shown that, when the equations of the non-holonomic constraints are specified by polynomials of order no higher than two with respect to the generalized velocities, the generalized forces of a system with energy dissipation comprise the sum of the gyroscopic, potential and dissipative forces.

In the numerical modelling of dynamical processes in systems with constraints,[1] it is necessary to take account of possible deviations from the equations of the constraints and to ensure stabilization of the constraints.[2] The general structure of the system of differential equations (SDE), the solutions of which satisfy the given equations of the constraints, is not uniquely defined.[3,4] The right-hand side of the required SDE contains arbitrary functions which determine the trajectories in the manifold corresponding to the equations of the constraints and the motion in the neighbourhood of this manifold.

From the set of SDEs, the solutions of which, with the appropriate initial conditions, satisfy the given equations of the constraints, it is possible to distinguish systems possessing additional properties. The conditions which ensure the stability of a manifold, the distribution of the family of trajectories in this domain and the required accuracy to which the equations of the constraints must be satisfied in the numerical solution are well-known.[4] Modern inverse problems in dynamics imply the construction of SDEs of a specific structure, by means of which an expression for the functional, which assumes a steady value for its solutions, is established. By virtue of well-known dynamical analogies, another physical systems can also possess the above mentioned properties.[1] Conditions have been obtained[5,6] under which an SDE reduces to the form of the Lagrange and Birkhoff equations and methods for determining the Lagrange and Birkhoff functions have been given. Problems of constructing Lagrange and Birkhoff functions in the case, when the number of equations of the constraints is the same as the dimension of the system, have been considered in Ref. 7.

This paper is concerned with solving the problem of constructing SDEs, the solutions of which satisfy specified relations, and representing them in the form of Lagrange equations. The conditions which ensure the stabilization of the constraints are determined.

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1. The construction of systems of differential equations

A material system, the kinematic state of which is determined by the phase coordinates $q^i; v^i = \dot{q}^j; \dot{q}^j = dq^j/dt\,(i, j = 1, \ldots, n)$, is considered. We shall assume that the dynamics of the material system are described by the following system of differential equations

$$\dot{q}^i = \upsilon^i, \quad a_{ij}(q^k, \upsilon^l, t)\dot{\upsilon}^j + a_i(q^k, \upsilon^l, t) = 0 \tag{1.1}$$

$$q^i(t_0) = q_0^i, \quad \upsilon^j(t_0) = \upsilon_0^j, \quad k, l = 1, \ldots, n \tag{1.2}$$

In equality (1.1) and henceforth, summation over common indices is assumed.

The following problem arises. It is required to determine the conditions which must be imposed on the functions a_{ij}, a_i in order that system (1.1) can be represented in the form of a Lagrange equation of the second kind

$$\begin{gathered}\dot{q}^i = \upsilon^i, \quad \frac{d}{dt}(\partial_i' L) - \partial_i L = 0\\ \partial_i'(\ ) = \partial(\ )/\partial\upsilon^i, \quad \partial_i(\ ) = \partial(\ )/\partial q^i\end{gathered} \tag{1.3}$$

and its solutions for all $t \geq t_0$ satisfy the equalities

$$\begin{gathered}f^\mu(q^i, t) = y^\mu, \quad f^{m_1+\mu} \equiv \partial_i f^\mu \upsilon^i + \partial_t f^\mu = y^{m_1+\mu}, \quad f^{m_1+\rho}(q^i, \upsilon^j, t) = y^{m_1+\rho}\\ \mu = 1, \ldots, m_1, \quad \rho = m_1 + 1, \ldots, m_1 + m_2 = m \leq n\end{gathered} \tag{1.4}$$

The functions y^μ, $y^{m_1+\mu}$, $y^{m_1+\rho}$ are treated as further variables, the values of which are determined by the equalities

$$\dot{y}^\pi = k_\theta^\pi(q^i, \upsilon^j, t)y^\theta; \quad y^\pi(t_0) = f^\pi(q_0^i, \upsilon_0^j, t_0), \quad \pi, \theta = 1, \ldots, m_1 + m \tag{1.5}$$

which are considered together with system (1.1). It follows from equalities (1.4) that the coefficients k_θ^μ of the equations of system (1.5) must be defined uniquely:

$$k_\mu^\nu = k_\sigma^\mu = 0, \quad \nu = 1, \ldots, m_1, \quad \sigma = 2m_1 + 1, \ldots, m_1 + m, \quad k_{m_1+\nu}^\mu = \delta_\nu^\mu$$

$$\delta_\nu^\mu = 0, \quad \mu \neq \nu, \quad \delta_\mu^\mu = 1$$

The rest of the coefficients remain arbitrary and can be used (see Refs. 2–4, for example) to solve the problem of stabilizing the constraints. If $y^\pi \equiv 0$ in equalities (1.4), they can be considered as the equations of the constraints imposed on the system, the dynamics of which is described by system (1.1) or (1.3).

In order to determine the structure of the functions a_{ij}, a_j, corresponding to the equations of the constraints (1.4) and (1.5), it is necessary to represent system (1.1) in a form which can be solved for the derivatives $\dot{v}^j$. This is possible if $\det(a_{ij}) \neq 0$, the inverse matrix (a^{jk}) exists and the elements of this matrix satisfy the conditions

$$a^{jk}a_{ki} = \delta_i^j, \quad \delta_i^i = 1, \quad \delta_i^j = 0, \quad i \neq j$$

System (1.1) then reduces to the form

$$\dot{q}^i = \upsilon^i, \quad \dot{\upsilon}^k = -a^k(q^i, \upsilon^j, t), \quad a^k = a^{kj}a_j \tag{1.6}$$

Differentiation of expressions (1.4), taking into account relations (1.5) and (1.6), leads to the equalities

$$f_i^\mu \upsilon^i + f_i^\mu = k_\theta^\mu y^\theta, \quad f_i^\mu \dot{\upsilon}^i + \partial_{ij}^2 f^\mu \upsilon^i \upsilon^j + 2\partial_{it}^2 f^\mu \upsilon^i + \partial_{tt}^2 f^\mu = k_\theta^{m_1+\mu} y^\theta \tag{1.7}$$

$$\begin{gathered}\partial_{ij}^2 f^\mu = \partial^2 f^\mu/\partial q^i \partial q^j, \quad \partial_{it}^2 f^\mu = \partial^2 f^\mu/\partial q^i \partial t, \quad \partial_{tt}^2 f^\mu = \partial^2 f^\mu/\partial t^2\\ \partial_j' f^\rho \dot{\upsilon}^j + \partial_i f^\rho \upsilon^i + \partial_t f^\rho = k_\theta^{m_1+\rho} y^\theta\end{gathered} \tag{1.8}$$

Substituting their expressions (1.6) into equalities (1.7) and (1.8) in place of $\dot{v}^i$, we obtain the following system of linear algebraic equations in the functions a_i

$$
\begin{gathered}
g^{\kappa i}a_i = h^{\kappa}, \quad \kappa = 1, \dots, m \\
g^{\mu i} = \partial_j f^{\mu} a^{ji}, \quad g^{\rho i} = \partial'_j f^{\rho} a^{ji} \\
h^{\mu} = \partial^2_{ij} f^{\mu} v^i v^j + 2\partial^2_{it} f^{\mu} v^i + \partial^2_{tt} f^{\mu} - k_{\theta}^{m_1+\mu} y^{\theta}, \quad h^{\rho} = \partial_i f^{\rho} v^i + \partial_t f^{\rho} - k_{\theta}^{m_1+\rho} y^{\theta}
\end{gathered}
\tag{1.9}
$$

The matrix $(g^{\kappa i})$ of system (1.9) is a rectangular $m \times n$ matrix. We will henceforth assume that the vectors $\mathbf{g}^{\kappa} = (g^{\kappa 1}, \dots, q^{\kappa n})$, which constitute the rows of the matrix $(g^{\kappa i})$, are linearly independent for all values of q^i, v^j, $t \geq t_0$. The general solution of system (1.9) is the sum of the two terms

$$
a_i = c a_i^{\tau} + a_i^{\nu}
\tag{1.10}
$$

the first of which is determined by the product of the arbitrary function $c = c(q^i, v^j, t)$ and the determinant $a_i^{\tau} = \det[(\delta_{ij}, g^{\kappa j}, c^{\xi j})]$, composed of the components of the unit vector $\delta^i = (\delta^{i1}, \dots, \delta^{in})$ $(\delta^{jj} = 1, \delta^{jk} = 0, k \neq j)$, the vectors $\mathbf{g}^{\kappa} = (g^{\kappa j})$ and the arbitrary vectors $\mathbf{c}^{\xi} = (c^{\xi 1}, \dots, c^{\xi n})$ $(\xi = m+1, \dots, n-1)$. The second term a_i^{ν} is determined by a linear combination of the rows of the matrix $(g^{\kappa i})$ with arbitrary coefficients λ_{κ}

$$
a_i^{\nu} = \delta_{ij}\lambda_{\kappa} g^{\kappa j}, \quad \delta_{ii} = 1, \quad \delta_{ij} = 0, \quad i \neq j
\tag{1.11}
$$

Substituting expression (1.10) into the left-hand side of Eq. (1.9), taking account of relation (1.11) and the equalities

$$
g^{\kappa i} a_i^{\tau} = 0, \quad g^{\kappa i}\delta_{ij} g^{\gamma j} = w^{\kappa\gamma}, \quad \gamma = 1, \dots, m
$$

we obtain a system of linear equations for determining the factors λ_{κ}

$$
w^{\gamma\kappa}\lambda_{\kappa} = h^{\gamma}
\tag{1.12}
$$

If the rows $\mathbf{g}^{\kappa}$ of the matrix $(g^{\kappa j})$ are linearly independent, then an inverse matrix $w_{\eta\gamma} : w_{\eta\gamma} w^{\gamma\kappa} = \delta^{\kappa}_{\eta}$ $(\eta = 1, \dots, m)$ exists and system (1.12) has the solution $\lambda_{\kappa} = w_{\kappa\gamma} h^{\gamma}$. This completes the proof of the following assertion.

Theorem 1. *If, in system (1.1), the functions a_i are defined by the expressions*

$$
a_i = c\det(\delta^{ij} g^{\kappa j} c^{\xi j}) + \lambda_{\kappa} g_i^{\kappa}, \quad \lambda_{\kappa} = w_{\kappa\gamma} h^{\gamma}
\tag{1.13}
$$

which contain the arbitrary functions c, $c^{\xi j}$ and $k_{\theta}^{m_1+\kappa}$, and the initial values (1.2) satisfy the conditions

$$
f^{\pi}(q_0^i, v_0^j, t_0) = 0
\tag{1.14}
$$

then, for the solutions $q^i = q^i(t)$, $v^j = v^j(t)$ corresponding to it, the following equalities are satisfied for all $t > t_0$

$$
f^{\pi}(q^i, v^j, t) = 0
\tag{1.15}
$$

In addition, by choosing the function $c^{\xi j}$, it is possible to control the motion of a representative point in the manifold defined by equalities (1.15) in the space of the variables q^i, v^j. The functions $k_{\theta}^{m_1+\kappa}$ affect the motion of a representative point in the neighbourhood of this manifold when conditions (1.14) are not satisfied (that is, when the right-hand sides of Eq. (1.14) are non-zero).

The conditions, imposed on the coefficients $k_{\theta}^{m_1+\kappa}$ of system (1.5) in order to ensure the asymptotic stability of the integral manifold of system (1.1), which is determined by Eq. (1.15), and the stabilization of the constraints in the numerical solution, have been determined in Ref. 4.

It can be seen from Eq. (1.13) that the components a_i^{τ} of the functions a_i contain arbitrary functions c, $c^{\xi j}$ and that the coefficients $k_{\theta}^{m_1+\kappa}$ occur in the expressions for the factors λ_{κ}. If the relation between the number n of equations of system (1.1) and the number n of equations of the constraints (1.4) permits, the functions c, $c^{\xi j}$, $k_{\theta}^{m_1+\kappa}$ can be chosen such that the expressions for a_i have a given structure.

The following three cases are possible.

1°. When $m = n$, system (1.9) contains n equations

$$g^{ki}a_i = h^k, \quad k = 1, \dots, n$$

and, for specific values of h^k, it has the unique solution

$$a_i = g_{ik}h^k, \quad g_{ik}g^{kj} = \delta_i^j$$

from which it follows that the functions a_i can have the required structure

$$a_i = p_i(q^j, \upsilon^k, t) \tag{1.16}$$

solely on account of the choice of the coefficients $k_\theta^{m_1+\kappa}$ on the right-hand sides of system (1.5).

2°. When $m = n - 1$, the solution of system (1.9) can be written in the form of (1.10), where $a_i^\tau = \det(\delta^{ij}g^{\kappa j})$. If it is required that the functions a_i should have the desired structure of (1.16), then the factor c and the coefficients $k_\theta^{m_1+\kappa}$ must be chosen such that the following equations are satisfied

$$ca_i^\tau + a_i^\nu = p_i \tag{1.17}$$

We now multiply both sides of Eq. (1.17) by $\delta^{ij}a_j^\tau$ and sum over the subscript i. Then, taking account of the identity $\delta^{ij}a_j^\tau a_i^\nu = 0$, we obtain an expression for the factor $c = \delta^{ij}a_j^\tau p_i/(\delta^{ij}a_j^\tau a_i^\tau)$, and Eq. (1.17) can be represented in the form of a system of linear equations which the functions $a_i = p_i(q^j, \upsilon^k, t)$ must satisfy

$$(b_k^i - \delta_k^i)p_i + a_k^\nu = 0, \quad b_k^i = \delta^{ij}a_k^\tau a_j^\tau/(\delta^{ij}a_i^\tau a_j^\tau) \tag{1.18}$$

Since the coefficients $(b_k^i - \delta_k^i)$ of system (1.18) are uniquely defined by Eqs. (1.1) and (1.4), then, in this case, only the coefficients $k_\theta^{m_1+\kappa}$ on the right-hand sides of system (1.5) have an effect on the structure of (1.16).

3°. When $m < n - 1$, the vector $\mathbf{a}^\tau = (a_1^\tau, \dots, a_n^\tau)$ is the vector product $\mathbf{a}^\tau = [\mathbf{g}^1 \dots \mathbf{g}^m \mathbf{c}^{m+1} \dots \mathbf{c}^{n-1}]$ of the vectors $\mathbf{g}^\mu = (g^{\mu j})$, $\mathbf{g}^\rho = (g^{\rho j})$ and the arbitrary vectors $\mathbf{c}^\xi = (c^{\xi j})$. We will use the notation $\mathbf{c}^{m+1} = \mathbf{g}^{m+1}, \dots, \mathbf{c}^{n-2} = \mathbf{g}^{n-2}$, $\mathbf{c}^{n-1} = \mathbf{c}$, put $h^{m+1}, \dots, h^{n-2}$ with arbitrary magnitudes and represent system (1.9) in the form

$$(\mathbf{g}^\kappa \mathbf{a}) = h^\kappa, \quad \kappa = 1, \dots, n-2 \tag{1.19}$$

Then, the general solution of system (1.19)

$$\mathbf{a} = c\mathbf{a}^\tau + \mathbf{a}^\nu, \quad \mathbf{a}^\tau = [\mathbf{g}^1 \dots \mathbf{g}^{n-2}\mathbf{c}], \quad \mathbf{a}^\nu = \lambda_\kappa \mathbf{g}^\kappa$$

depends on the choice of the coefficients $k_\theta^{m_1+\kappa}$, the arbitrary vector $\mathbf{c}$ and the scalar factor c and has the form

$$c\mathbf{g} + \lambda_\kappa \mathbf{g}^\kappa = \mathbf{p}, \quad \mathbf{g} = [\mathbf{g}^1 \dots \mathbf{g}^{n-2}\mathbf{c}], \quad \mathbf{a} = \mathbf{p}(q^i, \upsilon^j, t) \tag{1.20}$$

We now multiply both sides of equality (1.20) scalarly by the vector product $\mathbf{s} = [\mathbf{s}^0 \mathbf{s}^1 \dots \mathbf{s}^{n-2}]$ of the linearly independent vectors $\mathbf{s}^0, \mathbf{s}^1, \dots, \mathbf{s}^{n-2}$:

$$c(\mathbf{sg}) + \lambda_\kappa(\mathbf{sg}^\kappa) = (\mathbf{sp}) \tag{1.21}$$

The properties of the scalar, vector and skew vector products[8] enables us to use the following equalities

$$(\mathbf{sg}) = (\mathbf{s}^0[\mathbf{s}^1 \dots \mathbf{s}^{n-2}\mathbf{g}]) \tag{1.22}$$

$$(\mathbf{sg}) = \begin{vmatrix} (\mathbf{s}^0\mathbf{g}^1) & \dots & (\mathbf{s}^0\mathbf{g}^{n-2}) & (\mathbf{s}^0\mathbf{c}) \\ \dots & \dots & \dots & \dots \\ (\mathbf{s}^{n-2}\mathbf{g}^1) & \dots & (\mathbf{s}^{n-2}\mathbf{g}^{n-2}) & (\mathbf{s}^{n-2}\mathbf{c}) \end{vmatrix} \tag{1.23}$$

Using the second equality of (1.20) and equality (1.22), expression (1.23) can be represented in the form

$$(\mathbf{s}^0[\mathbf{s}^1\dots\mathbf{s}^{n-2}\mathbf{g}]) = \mathbf{s}^0\mathbf{g}, \quad \mathbf{g} = \begin{vmatrix} \mathbf{g}^1 & \dots & \mathbf{g}^{n-2} & \mathbf{c} \\ (\mathbf{s}^1\mathbf{g}^1) & \dots & (\mathbf{s}^1\mathbf{g}^{n-2}) & (\mathbf{s}^1\mathbf{c}) \\ \dots & \dots & \dots & \dots \\ (\mathbf{s}^{n-2}\mathbf{g}^1) & \dots & (\mathbf{s}^{n-2}\mathbf{g}^{n-2}) & (\mathbf{s}^{n-2}\mathbf{c}) \end{vmatrix}$$

Since the vector $\mathbf{s}^0$ is arbitrary, it follows from the last equality that

$$[\mathbf{s}^1\dots\mathbf{s}^{n-2}\mathbf{g}] = \begin{vmatrix} \mathbf{g}^1 & \dots & \mathbf{g}^{n-2} & \mathbf{c} \\ (\mathbf{s}^1\mathbf{g}^1) & \dots & (\mathbf{s}^1\mathbf{g}^{n-2}) & (\mathbf{s}^1\mathbf{c}) \\ \dots & \dots & \dots & \dots \\ (\mathbf{s}^{n-2}\mathbf{g}^1) & \dots & (\mathbf{s}^{n-2}\mathbf{g}^{n-2}) & (\mathbf{s}^{n-2}\mathbf{c}) \end{vmatrix} \tag{1.24}$$

Taking account of relations (1.22)–(1.24), we shall consider equality (1.21) as an equation in the vector **c**:

$$[\mathbf{s}^1\dots\mathbf{s}^{n-2}(c\mathbf{g} + \lambda_\kappa\mathbf{g}^\kappa - \mathbf{p})] = 0 \tag{1.25}$$

If the vectors $\mathbf{s}^1, \dots, \mathbf{s}^{n-2}$ and **c** satisfy the conditions

$$(\mathbf{s}^1\mathbf{c}) = 0, \dots, (\mathbf{s}^{n-2}\mathbf{c}) = 0$$

it then follows from Eq. (1.25) that

$$\mathbf{c} = \frac{1}{c\Gamma}[\mathbf{s}^1\dots\mathbf{s}^{n-2}(\mathbf{p} - \lambda_\kappa\mathbf{g}^\kappa)], \quad \Gamma = \begin{vmatrix} (\mathbf{s}^1\mathbf{g}^1) & \dots & (\mathbf{s}^1\mathbf{g}^{n-2}) \\ \dots & \dots & \dots \\ (\mathbf{s}^{n-2}\mathbf{g}^1) & \dots & (\mathbf{s}^{n-2}\mathbf{g}^{n-2}) \end{vmatrix} \tag{1.26}$$

It is easily verified that substitution of expression (1.26) for the vector **c** into the equality (1.20) converts it into an identity.

Example 1. We will now consider the problem of determining the angular velocity vector $\boldsymbol{\omega}$ of a rigid body using the known velocities of three of its points which do not lie on one straight line. Suppose $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ are the radius vectors of the points and that

$$\boldsymbol{\rho}_1 \neq k\boldsymbol{\rho}_2, \quad \boldsymbol{\rho}_1 = \mathbf{r}_2 - \mathbf{r}_1, \quad \boldsymbol{\rho}_2 = \mathbf{r}_3 - \mathbf{r}_2, \quad \boldsymbol{\rho}_3 = \mathbf{r}_1 - \mathbf{r}_3$$

$$\mathbf{v}_1 = \dot{\mathbf{r}}_1, \quad \mathbf{v}_2 = \dot{\mathbf{r}}_2, \quad \mathbf{v}_3 = \dot{\mathbf{r}}_3$$

The equation $(\boldsymbol{\rho}_i\dot{\boldsymbol{\rho}}_i) = 0$ for determining $\boldsymbol{\rho}_i$ then follows from the equalities $(\boldsymbol{\rho}_i\boldsymbol{\rho}_i) = \rho_i^2 = \text{const}\,(i = 1, 2, 3)$ and the solution of this equation is $\dot{\boldsymbol{\rho}}_i = c_i\,[\boldsymbol{\rho}_i\boldsymbol{\omega}_i]$, where c_i is an arbitrary scalar and $\boldsymbol{\omega}_i$ is an arbitrary vector. Then, by putting $c_i = -1$ and differentiating the scalar product $(\boldsymbol{\rho}_i\boldsymbol{\rho}_j) = \rho_{ij} = \text{const}\,(i \neq j)$, we arrive at the equality

$$\boldsymbol{\omega}_{ij}[\boldsymbol{\rho}_i\boldsymbol{\rho}_j] = 0; \quad \boldsymbol{\omega}_{ij} = \boldsymbol{\omega}_i - \boldsymbol{\omega}_j$$

from which it follows that $\boldsymbol{\omega}_i = \boldsymbol{\omega}_j = \boldsymbol{\omega}$ and

$$[\boldsymbol{\rho}_1\boldsymbol{\omega}] = \mathbf{v}_{12}, \quad [\boldsymbol{\rho}_2\boldsymbol{\omega}] = \mathbf{v}_{23}, \quad [\boldsymbol{\rho}_3\boldsymbol{\omega}] = \mathbf{v}_{31}, \quad \mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j \tag{1.27}$$

Multiplying equalities (1.27) vectorially on the left by $\mathbf{v}_3$, $\mathbf{v}_1$, $\mathbf{v}_2$ and adding, we obtain

$$(\mathbf{v}_3\boldsymbol{\omega})\boldsymbol{\rho}_1 + (\mathbf{v}_1\boldsymbol{\omega})\boldsymbol{\rho}_2 + (\mathbf{v}_2\boldsymbol{\omega})\boldsymbol{\rho}_3 - ((\mathbf{v}_3\boldsymbol{\rho}_1) + (\mathbf{v}_1\boldsymbol{\rho}_2) + (\mathbf{v}_2\boldsymbol{\rho}_3))\boldsymbol{\omega} =$$
$$= 2([\mathbf{v}_2\mathbf{v}_1] + [\mathbf{v}_1\mathbf{v}_2] + [\mathbf{v}_2\mathbf{v}_3])$$

Taking account of the obvious equalities

$$\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2 + \boldsymbol{\rho}_3 = 0, \quad (\mathbf{v}_{ij}\boldsymbol{\omega}) = 0$$

we arrive at the conclusion that

$$(\mathbf{v}_3\boldsymbol{\omega})\boldsymbol{\rho}_1 + (\mathbf{v}_1\boldsymbol{\omega})\boldsymbol{\rho}_2 + (\mathbf{v}_2\boldsymbol{\omega})\boldsymbol{\rho}_3 = 0$$

and obtain an expression for the angular velocity of the rigid body[9]

$$\boldsymbol{\omega} = 2([\mathbf{v}_3\mathbf{v}_1] + [\mathbf{v}_1\mathbf{v}_2] + [\mathbf{v}_2\mathbf{v}_3])((\mathbf{v}_3\boldsymbol{\rho}_1) + (\mathbf{v}_1\boldsymbol{\rho}_2) + (\mathbf{v}_2\boldsymbol{\rho}_3))^{-1}$$

2. Non-holonomic preset constraints of the first and second powers of the generalized velocities

We will now consider a special case of the problem of determining the coefficients a_{ij}, a_i of the equations of system (1.1), when $m_1 = 0$ and the functions $f^{m_1+\rho}(q^i, v^j, t) = y^{m_1+\rho}$ contain generalized velocities v^j to a power no higher than the second

$$f^\mu = f_i^\mu(q^k)v^i + f_0^\mu(q^k), \quad \mu = 1, \dots, m$$
$$f^{m+1} = \frac{1}{2}f_{ij}^{m+1}(q^k)v^iv^j + f_i^{m+1}(q^k)v^i + f_0^{m+1}(q^k), \quad f_{ij}^{m+1} = f_{ji}^{m+1}$$

We note that the equations of the constraints of the form $f_i^\mu(q^k)v^i + f_0^\mu(q^k) = 0$ may be integrable. Putting $a_{ij} = f_{ji}^{m+1}$, from the relations

$$\dot{f}^\kappa = F^\kappa(f^\pi, q^i, v^j)$$

where $F^\kappa(f^\pi, q^i, v^j)$ $(F^\kappa(0, q^i, v^j) = 0$ are arbitrary functions, $\kappa, \pi = 1, \dots, m+1$, we obtain a system of equations for determining the functions a_l

$$\begin{aligned} &g^{\kappa l}a_l = h^\kappa, \quad l = 1, \dots, n \\ &g^{\mu l} = f_i^\mu a^{il}, \quad g^{(m+1)l} = v^l + r^l, \quad r^l = f_i^{m+1}a^{il}, \quad a_{ki}a^{il} = \delta_k^l \\ &h^\mu = \partial_k f_i^\mu v^iv^k + \partial_k f_0^\mu v^k - F^\mu \\ &h^{m+1} = \frac{1}{2}\partial_k f_{ij}^{m+1}v^iv^jv^k + \partial_k f_i^{m+1}v^iv^k + \partial_k f_0^{m+1}v^k - F^{m+1} \end{aligned} \tag{2.1}$$

The component a_l^τ of the solution of system (2.1) is determined in the form

$$a_l^\tau = a_{lj}^\tau(v^j + r^j)$$

where

$$a_{jl}^{\tau} = -a_{lj}^{\tau} = \begin{vmatrix} \delta^{j1} & \dots & \delta^{jj} & \dots & \delta^{jl} & \dots & \delta^{jn} \\ \delta^{l1} & \dots & \delta^{lj} & \dots & \delta^{ll} & \dots & \delta^{ln} \\ g^{11} & \dots & g^{1j} & \dots & g^{1l} & \dots & g^{1n} \\ . & \dots & . & \dots & . & \dots & . \\ g^{(n-2)l} & \dots & g^{(n-2)j} & \dots & g^{(n-2)l} & \dots & g^{(n-2)n} \end{vmatrix} \tag{2.2}$$

If the arbitrary functions $g^{(m+1)j}, \dots, g^{(n-2)j}$, the components of the last rows of the determinant, depend solely on the coordinates q^i, then $a_{lj}^{\tau} = a_{lj}^{\tau}(q^i)$ and the following condition is satisfied

$$a_{lj}^{\tau}(q^i)\upsilon^l\upsilon^j = 0$$

3. A system with energy dissipation

It is required to construct equations of the dynamics

$$\dot{q}^i = \upsilon^i, \quad a_{ij}(q^k, \upsilon^l, t)\dot{\upsilon}^j + a_i(q^k, \upsilon^l, t) = 0 \tag{3.1}$$

of a system, for which the following equality is satisfied

$$\begin{gathered} \dot{E} = -R \\ E = T + V,\ T = \frac{1}{2}m_{ij}(q^k)\upsilon^i\upsilon^j,\ m_{ij} = m_{ji},\ V = V(q^k),\ R = \frac{1}{2}b_{ij}(q^k)\upsilon^i\upsilon^j,\ b_{ij} = b_{ji} \end{gathered} \tag{3.2}$$

We put $a_{ij} = m_{ij}(q^k)$. Then, the equation for determining the functions a_i

$$\begin{gathered} \upsilon^i(\partial_i V + \gamma_{i,jk}\upsilon^j\upsilon^k + b_{ij}\upsilon^j - a_i) = 0 \\ \gamma_{i,jk} = \frac{1}{2}(\partial_j m_{ki} + \partial_k m_{ij} - \partial_i m_{jk}) \end{gathered} \tag{3.3}$$

follows from relations (3.1) and (3.2).

The set of solutions of Eq. (3.3) is written in the form

$$a_l = a_{lj}^{\tau}\upsilon^j + \partial_l V + b_{lj}\upsilon^j + \gamma_{lj,k}\upsilon^j\upsilon^k$$

The coefficients a_{jl}^{τ} are determined by expressions (2.2) with the arbitrary functions $g^{1j}, \dots, g^{(n-2)j}$. In the case being considered, system (1.1) takes the form

$$\dot{q}^j = \upsilon^j, \quad m_{ij}\dot{\upsilon}^j + \gamma_{ij,k}\upsilon^j\upsilon^k = -a_{ij}^{\tau}\upsilon^j - \partial_i V - b_{ij}\upsilon^j \tag{3.4}$$

The right-hand side of the system of equations (3.4) consists of the sum of the gyroscopic, potential and dissipative forces.

4. Helmholtz conditions

In order that the system (1.1) can be represented in the form of Lagrange equations, the functions a_{ij} and a_i must satisfy the Helmholtz conditions[10]

$$\begin{gathered} a_{ij} = a_{ji}, \quad \partial'_k a_{ij} = \partial'_i a_{kj}, \quad \partial'_k a_i + \partial'_i a_k = 2\{\partial_t + \upsilon^j\partial_j\}a_{ik} \\ 2(\partial_j a_i - \partial_i a_j) = \{\partial_t + \upsilon^k\partial_k\}(\partial'_j a_i - \partial'_i a_j) \end{gathered} \tag{4.1}$$

Suppose the coefficients a_{ij} of system (1.1) have the form

$$a_{ij} = a_{ij,0}(q^l) + a_{ij,k}(q^l)\upsilon^k + a_{ij}^{(2)}(q^k, \upsilon^l) \tag{4.2}$$

and that the functions a_i with an appropriate choice of the arbitrary functions $c, c^{\xi j}, k_\theta^\kappa$ can be represented by the an expansion in powers of υ^i:

$$a_i = a_{i,0}(q^l) + a_{i,j}(q^l)\upsilon^j + \frac{1}{2}a_{i,jk}(q^l)\upsilon^i\upsilon^k + a_i^{(3)}(q^k, \upsilon^l) \tag{4.3}$$

The coefficients $a_{i,jk}$ on the right-hand side of Eq. (4.3) satisfy the conditions $a_{i,jk} = a_{i,kj}$; $a_{ij}^{(2)}(q^k, \upsilon^l)$, $a_i^{(3)}(q^k, \upsilon^l)$ in expressions (4.2) and (4.3) respectively denote terms containing υ^i to powers of no less than the second and third. Equalities (4.1) then lead to the following conditions which are imposed on the coefficients of expansions (4.2) and (4.3) with respect to the powers of υ^i

$$\begin{aligned}
&a_{ij,0} = a_{ji,0}, \quad a_{ij,k} = a_{ji,k} = a_{kj,i}\\
&a_{i,k} + a_{k,i} = 0, \quad \partial_j a_{i,0} = \partial_i a_{j,0}, \quad \partial_i a_{j,k} + \partial_j a_{k,i} + \partial_k a_{i,j} = 0\\
&a_{i,jk} + a_{k,ji} = 2\partial_j a_{ki,0}, \quad \partial_j a_{i,kl} - \partial_i a_{j,kl} = \partial_k(a_{i,jl} - a_{j,il})\\
&a_{ij}^{(2)} = a_{ji}^{(2)}, \quad \partial'_k a_{ij}^{(2)} = \partial'_i a_{kj}^{(2)}\\
&\partial'_k a_i^{(3)} + \partial'_i a_K^{(3)} - 2\partial_j a_{ik,l}\upsilon^j\upsilon^l = 2\upsilon^j\partial_j a_{ik}^{(2)}\\
&2(\partial_j a_i^{(3)} - \partial_i a_j^{(3)}) = \partial_k(\partial'_j a_i^{(3)} - \partial'_i a_j^{(3)})\upsilon^k
\end{aligned} \tag{4.4}$$

Hence, the following assertion holds.

Theorem 2. *System* (1.1) *with coefficients of the form of* (4.2) and (4.3) *reduces to the form of the Lagrange* Eq. (1.3) *if conditions* (4.4) *are satisfied.*

It has been shown[5,7] that the Lagrangian, corresponding to system (1.3), can be sought in the form

$$L = k(q^i, \upsilon^j) + d_j(q^i)\upsilon^j + d_0(q^i) \tag{4.5}$$

where the functions $k(q^i, \upsilon^j)$, $d_j(q^i)$, $d_0(q^i)$ satisfy the partial differential equations

$$\partial'_i\partial'_j k = a_{ij} \tag{4.6}$$

$$\partial_j d_i - \partial_i d_j = (\partial'_j a_i - \partial'_i a_j)/2 + (\partial_i\partial'_j k - \partial_j\partial'_i k) \equiv z_{ij}(q^l) \tag{4.7}$$

$$\partial_i d_0 = -a_i - \partial_i k + (\partial_i\partial'_j k + (\partial_j a_i - \partial_i a_j)/2)\upsilon^j \equiv w_i(q^l) \tag{4.8}$$

The solution of the equations of system (4.6)–(4.8) has been given in Ref. 5

Example 2. Suppose the kinematic state of a mechanical system is determined by the coordinates

$$q^1 = x, \quad q^2 = y, \quad q^3 = \dot{x}, \quad q^4 = \dot{y}$$

which must satisfy the conditions

$$f = \dot{x}^2 + \dot{y}^2 + 2V(x, y) - 2e, \quad e = \text{const}, \quad \dot{f} = \lambda f \tag{4.9}$$

System (1.1) takes the form

$$a_{11}\ddot{x} + a_{12}\ddot{y} + a_1 = 0, \quad a_{21}\ddot{x} + a_{22}\ddot{y} + a_2 = 0 \tag{4.10}$$

Putting

$$a_{11} = a_{22} = 1, \quad a_{12} = a_{21} = 0, \quad a_{ij}^{(2)} = 0, \quad i, j = 1, 2$$

from equalities (4.9) and (4.10), we obtain an equation for determining the functions a_1, a_2

$$\dot{x}(\partial_x V - a_1) + \dot{y}(\partial_y V - a_2) = pf; \quad \partial_x(\ ) = \partial(\ )/\partial x, \quad \partial_y(\ ) = \partial(\ )/\partial y \tag{4.11}$$

The solution of Eq. (4.11)

$$a_1 = \partial_x V + c\dot{y} + p\dot{c}f/\upsilon^2, \quad a_2 = \partial_y V - c\dot{x} + p\dot{y}f/\upsilon^2; \quad \upsilon^2 = \dot{x}^2 + \dot{y}^2$$

contains the arbitrary functions $c(x, y, \dot{x}, \dot{y})$ and $p(x, y, \dot{x}, \dot{y})$, by the choice of which the structure of the equations of system (4.10) is determined. We put $p = -\lambda v^2$. If the factors c and λ are independent of the generalized velocities $\dot{x}, \dot{y} : c = c(x, y), \lambda = \lambda(x, y)$, then

$$a_i = a_{i,0} + a_{i,1}\dot{x} + a_{i,2}\dot{y} + (a_{i,11}\dot{x}^2 + 2a_{i,12}\dot{x}y + a_{i,22}\dot{y}^2)/2 + a_i^{(3)}$$

$$a_{1,0} = \partial_x V, \quad a_{2,0} = \partial_y V, \quad a_{1,1} = -a_{2,2} = -2\lambda(e - V), \quad a_{1,2} = -a_{2,1} = -c$$

$$a_{i,jk} \equiv 0, \quad a_1^{(3)} = \lambda\dot{x}\upsilon^2, \quad a_2^{(3)} = \lambda\dot{y}\upsilon^2$$

Direct calculations show that conditions (4.4) are satisfied when $\lambda \equiv 0$. System (4.10) is written in the form

$$\ddot{x} + \partial_x V(x, y) + c(x, y)\dot{y} = 0, \quad \ddot{y} + \partial_y V(x, y) - c(x, y)\dot{x} = 0 \tag{4.12}$$

We now construct the Lagrangian corresponding to system (4.12). The function $k = (\dot{x}^2 + \dot{y}^2)/2$, corresponding to system (4.12), is determined from Eqs. (4.6) and (4.7) are written in the form

$$\partial_y V d_1 - \partial_x V d_2 = c(x, y) \tag{4.13}$$

The arbitrary functions $d_1 = \alpha_1(x, y)$, $d_2 = \alpha_2(x, y)$ satisfy Eq. (4.13). Eq. (4.8) for determining the function d_0 takes the form

$$\partial_x d_0 = -\partial_x V, \quad \partial_y d_0 = -\partial_y V$$

whence it follows that $d_0 = -V(x, y)$. Substituting the values which have been found for k, d_0, d_1, d_2 into equality (4.5), we obtain the Lagrange function

$$L = (\dot{x}^2 + \dot{y}^2)/2 + \dot{x}\alpha_1(x, y) + \dot{y}\alpha_2(x, y) - V(x, y) + e$$

which, when $e = 0$, is identical to the function L constructed earlier in Ref. 11.

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